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Berezin integrals and Poisson processes

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Abstract. We show that the calculation of Berezin integrals over anticommuting variables can be reduced to the evaluation of expectations of functionals of Poisson processes via an appropriate Feynman–Kac formula. In this way the tools of ordinary analysis can be applied to Berezin integrals and, as an example, we prove a simple upper bound. Possible applications of our results are briefly mentioned.

Dedicated to the memory of Michel Sirugue

1. Introduction

In the last few decades the functional integral has become the standard approach to the quantization of systems with infinitely many degrees of freedom. In typical cases like QED and QCD which involve both bosons and fermions, the integral has to deal with anticommuting variables belonging to a Grassmann algebra and a lucid exposition of the rules of integration over these variables can be found in [B1, FS].

With the discovery of supersymmetry (SUSY) anticommutative integration received further impetus and has been applied in different areas of physics and mathematics. SUSY was first introduced in particle physics to express a possible fundamental symmetry between bosons and fermions and then found several applications as a formal tool in the theory of complex systems like heavy nuclei and more recently disordered or chaotic mesoscopic systems [VWZ, F, Z]. In mathematics it plays a role in various approaches to index theorems and other topics of differential geometry [BZ].

In physical applications of SUSY one introduces fields Ψ which have two types of components, boson-like $\varphi_1, \ldots, \varphi_{2n}$ which are just ordinary numerical variables and fermi-like $\xi_1, \ldots, \xi_{2n}$ where the ξ's are anticommuting Grassmann variables. Supersymmetry transformations mix boson and fermion components.

One is usually interested in the calculation of formal expressions like

$$\int d[\phi]\, d[\xi]\, \prod_i \varphi_i \prod_j \xi_j \exp(-S(\Psi))$$

called correlation functions. $S(\Psi)$ is a functional invariant under supersymmetry transformations.

Integrals over anticommuting variables were also introduced by Berezin in statistical mechanics to represent the partition function of the planar Ising model [B2] and the generating function of related combinatorial problems [B3]. See also [RZ].

The rules for the calculation of these integrals, known as Berezin integrals, are recalled in section 2. In spite of the fact that this formalism allows compact and very powerful manipulations of the expressions of interest, and in some cases exact calculations, it has the drawback that the usual tools of analysis are not applicable to the anticommuting variables. In particular one cannot easily find bounds for anticommutative integrals.

The purpose of this paper is to show that any Berezin integral can be represented in terms of the expectations of appropriate functionals of Poisson processes. On the basis of this representation, ordinary analysis can be used and in particular upper bounds can be obtained. Furthermore, correlation functions such as those above can be expressed entirely as expectations over ordinary stochastic variables.

The starting point of our analysis is a generalized Feynman–Kac formula developed in the early 1980s to express the solutions of the imaginary-time Pauli equation [DJLS]. The equation is

$$\partial_t \psi_t = -\tfrac{1}{2}(-i\nabla - \boldsymbol{A}(x))^2 \psi_t - V(x)\psi_t + \tfrac{1}{2}\boldsymbol{H}(x)\cdot\boldsymbol{\sigma}\psi_t \tag{1.1}$$

where $\boldsymbol{A}$ and $\boldsymbol{H}$ are the vector potential and the magnetic field respectively. $\boldsymbol{\sigma}$ denotes the Pauli matrices in the usual representation. In [DJLS] we proved that the initial value problem is solved by

$$\begin{aligned}\psi_t(x,\sigma) = \mathrm{e}^t\, \mathbf{E}\Bigg[\psi_0(x+w_t, (-)^{N_t}\sigma)\exp\bigg(&-\int_0^t V(x+w_\tau)\,d\tau - i\int_0^t \boldsymbol{A}(x+w_\tau)\cdot d\mathbf{w}_\tau \\ &+\frac{1}{2}\int_0^t H_z(x+w_\tau)(-)^{N_\tau}\sigma\,d\tau + \int_0^t \log\bigg[\frac{1}{2}\big(H_x(x+w_\tau) \\ &-i(-)^{N_\tau}\sigma H_y(x+w_\tau)\big)\bigg]\,dN_\tau\bigg)\Bigg].\end{aligned} \tag{1.2}$$

Here σ is a dichotomic variable which can take the values ± 1. The expectation is taken with respect to the Wiener process w_t and the Poisson process N_t. To understand this formula we have to explain the meaning of the stochastic integral $\int dN_t$. A Poisson process is a jump process characterized by the following probabilities:

$$P(N_{t+\Delta t} - N_t = k) = \frac{(\Delta t)^k}{k!}\mathrm{e}^{-\Delta t}.$$

Its trajectories are therefore piecewise constant increasing functions and we shall assume that at each jump they are continuous from the left. The stochastic integral is just an ordinary Stieltjes integral

$$\int_0^t f(\tau)\,dN_\tau = \sum_1^n f(\tau_i)$$

where τ_i are random jump times in the interval $[0,t)$ which are distributed exponentially, i.e. $P(\tau \leqslant t) = 1 - \mathrm{e}^{-t}$.

The interesting property of equation (1.2) is that by letting the spinor indices to become a stochastic process the Pauli matrices have disappeared from the expression of the evolution operator and their algebra is completely taken into account by the expectation over the jump process. The power of the approach was demonstrated by proving a non trivial paramagnetic inequality which shows that in three dimensions the evolution is bounded above by the evolution in a magnetic field which lies in a plane and whose components are simply related to the original magnetic field. This can easily be seen by taking absolute values and implies for the ground states

$$E_0(0, 0, 0; (H_1^2 + H_2^2)^{1/2}, 0, H_3) \leqslant E_0(A_1, A_2, A_3; H_1, H_2, H_3).$$

For a recent application of the [DJLS] approach see [ER].

Since the Pauli matrices are objects which belong to a Clifford algebra the above findings suggested a possible connection between calculus with Poisson processes and calculus with anticommuting variables. It is the main purpose of this paper to implement such a connection. To illustrate how the connection comes about, in section 3 we again take up the case of Pauli-type equations and we observe that they can be interpreted as evolution equations over a Grassmann algebra. This type of evolution was considered, for instance, by Berezin and Marinov [BM, M]. Solutions of evolution equations over Grassmann algebras can be expressed as Berezin integrals which represent the convolution of the kernel of the evolution operator with the initial condition. A straightforward comparison with the solution given in [DJLS] provides the identification of the anticommutative integrals with the appropriate Poisson expectations.

In order to develop the theory in a systematic way for an arbitrary but finite number of anticommutative variables, in section 4 we introduce a representation of Grassmann algebras in the space of functions of dichotomic variables which we call the σ-representation. This space was used by Wigner in his book on group theory [W] to find the representations of the symmetric group connected with the exclusion principle.

In section 5 we develop the necessary theory of semigroups associated with Poisson processes and in section 6 we identify the Berezin integrals with the appropriate expectations. We also derive a general inequality.

In section 7 we discuss in particular the Gaussian anticommutative integrals due to their importance in physical applications.

We conclude this introduction with some comments on possible interesting applications of the results obtained in this paper. The semigroups associated to Poisson processes encompass all Hamiltonian semigroups $\{\exp(-tH)\}_{t\geqslant 0}$ for k interacting $\frac{1}{2}$-spins in an external magnetic field as in the case, for instance, of a Heisenberg ferromagnet or, by a slight change of language, of any Hamiltonian semigroup for models describing interacting fermions on a finite lattice. Our representation can therefore be used both for theoretical or simulation studies of the statistical mechanics of such systems.

A particularly interesting case to which our approach can be applied is the calculation of the Dirac propagator on a lattice, an important problem in the study of QCD on the lattice. Presumably in this way it is possible to obtain a simplification of the methods used at present in simulations. See the remark at the end of section 7.

More generally in all cases where SUSY is relevant our approach may be a useful tool.

We believe that our results are also of interest in their own right, insofar as they establish a direct connection between algebraic objects, such as those represented by Berezin integrals, and analytic expressions.

As a final remark we observe that the Wiener process and the Poisson process are both Levy processes and are actually the two limiting cases of the Levy–Khinchin formula. We therefore find their correspondence at the Euclidean level with the two basic types of particles in nature, namely bosons and fermions, quite satisfying. Then the question naturally arises: do the other processes described by the Levy–Khinchin formula have any relevance for physics?

2. Analysis on Grassmann algebras

We open this section by a short review of standard definitions and results and refer the reader to [B1, D, FS] for more detailed information.

We denote by $\mathbf{G}(k)$ the Grassmann algebra over $\mathbb{C}$ generated by its identity $\mathbf{1}_k$ and a family $\{\xi_1, \ldots, \xi_k\}$ of generators which obey the following commutation relations:

$$\xi_i \xi_j = -\xi_j \xi_i \qquad \forall\, i, j. \tag{2.1}$$

For future reference, $\mathbf{G}^n(k) \simeq \mathbf{G}(nk)$ will be the Grassmann algebra generated by $\mathbf{1}_k^n$ and $\{\xi_1^1, \ldots, \xi_k^1, \xi_1^2, \ldots, \xi_k^2, \ldots, \xi_1^n, \ldots, \xi_k^n\}$, and by convention, $\mathbf{G}^1(k) = \mathbf{G}(k)$.

The collection $\{1, \ldots, k\}$ of labels is any non-empty finite set endowed with a total ordering.

Elements of $\mathbf{G}(k)$ of the form $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}$ are called monomials; we will use the set of ordered multi-indices $M_k = \{\mu = (\mu_1, \ldots, \mu_n) : 1 \leqslant \mu_1 < \mu_2 < \cdots < \mu_n \leqslant k\}$, and write

$$\xi^\mu = \xi_{\mu_1} \cdots \xi_{\mu_n}$$

for $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. As a linear space $\mathbf{G}(k)$ has dimension 2^k, and each element $F(\xi) \in \mathbf{G}(k)$ can be represented in a unique way as a polynomial with complex coefficients:

$$F(\xi) = F(\xi) = f_0 \cdot \mathbf{1}_k + \sum_{r=1}^{k} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant k} f_{i_1,\ldots,i_k} \xi_{i_1} \cdots \xi_{i_r} = \sum_{\mu \in M_n} f_\mu \cdot \xi^\mu \tag{2.2}$$

where $f_\mu \in \mathbb{C}$, and therefore $\mathbf{G}(k)$ is naturally graded. It is advantageous to think of $F(\xi)$ as a 'function' of the Grassmann 'variables' $\xi_1, \ldots, \xi_k$, that is to say of the 'Fermi field' $\{\xi_1, \ldots, \xi_k\}$.

Analysis over $\mathbf{G}(k)$ is based upon left derivatives $\delta/\delta\xi_1, \ldots, \delta/\delta\xi_k$ and the Berezin integral, which are defined as follows:

$$\begin{aligned} \frac{\delta}{\delta\xi_i} \xi_{\mu_1} \cdots \xi_{\mu_n} &\stackrel{\text{def}}{=} \delta_{\mu_1 i} \cdot \xi_{\mu_2} \cdots \xi_{\mu_n} - \delta_{\mu_2 i} \cdot \xi_{\mu_1} \xi_{\mu_3} \cdots \xi_{\mu_n} \\ &\quad + (-1)^{k-1} \delta_{\mu_k i} \cdot \xi_{\mu_1} \cdots \xi_{\mu_{k-1}} \\ &= \sum_{j=1}^{k} (-1)^{j-1} \delta_{\mu_j i} \cdot \xi_{\mu_1} \cdots \cancel{\xi}_{\mu_j} \cdots \xi_{\mu_{k-1}} \end{aligned}$$

where the / sign through the generator ξ_{μ_j} means that it is omitted.

To define an integral we introduce symbols $d\xi_1, \ldots, d\xi_k$ satisfying the following commutation relations:

$$\{d\xi_i, d\xi_j\} = \{d\xi_i, \xi_j\} = 0$$

where $\{a, b\} = a \cdot b + b \cdot a$, and define 'basic' integrals

$$\int^B d\xi_i = 0 \qquad \int^B \xi_\mu \cdot \xi_i \; d\xi_i = \xi_\mu$$

if $\mu_j \neq i$ for all j in μ. We extend the integral on $\mathbf{G}(k)$ by linearity and call it the Berezin integral. In general

$$\int^B F(\xi)\; d\xi_k \cdots d\xi_1 \stackrel{\text{def}}{=} \int^B F(\xi)\; \mathcal{D}_k\xi \equiv f_{1,\ldots,k} \in \mathbb{C} \tag{2.3}$$

and observe that up to a multiplicative constant, is uniquely defined as the only linear form over $\mathbf{G}(k)$ alternating under permutations of the Grassmannian variables. It transforms [B1] as

$$\int^B F(R\xi)\,\mathcal{D}_k\xi = (\det R)\cdot\int^B F(\xi)\,\mathcal{D}_k\xi \tag{2.4}$$

under the linear substitution $\xi_i \to \sum_{j=1}^k R_{ij}\xi_j$.

An especially important case is provided by 'Gaussian integrals':

$$\int^B \exp\left\{\frac{1}{2}\sum_{i,j=1}^k A_{ij}\xi_i\xi_j\right\}\mathcal{D}_k\xi \tag{2.5}$$

where one can always assume that $A = (A_{ij})$ is an antisymmetric matrix, otherwise it could be replaced by $2^{-1}(A - A^{\mathrm{T}})$. 'Gaussian integrals' vanish for k odd while, by exploiting (2.4), we obtain

$$\int^B \exp\left\{\sum_{1\leqslant h<k\leqslant 2k} A_{hk}\xi_h\xi_k\right\}\mathcal{D}_{2k}\xi = Pf\ A. \tag{2.6}$$

The Pfaffian $Pf\ A$ of the triangular array $\{A_{hk}\}_{1\leqslant h<k\leqslant 2k}$ is defined by

$$Pf\,A = \sum_\pi (-1)^\pi A_{i_1 j_1}\cdots A_{i_k j_k} \tag{2.7}$$

where the sum $\sum_\pi$ is taken over all $(2k)!/2^k k!$ ways of pairing of the elements of the set $\{1, 2, \ldots, 2k\}$ where $(-1)^\pi$ is the parity of the permutation $\pi = (i_1 j_1, \ldots, i_k j_k)$.

Let us consider the operator of left differentiation $\delta/\delta\xi_i$ and the operator of left multiplication $\widehat{\xi_i}$ by the element ξ_i, both acting on $\mathbf{G}(k)$ (see [B1]). In what follows we shall omit the hat if confusion does not arise.

We recall that all linear operators acting on $\mathbf{G}(k)$ belong to the Clifford (or Spinor) algebra $\mathbf{C}(2k)$ generated by its identity $\hat{\mathbf{1}}_{2k}$ and the operators $\delta/\delta\xi_1, \ldots, \delta/\delta\xi_k$, and $\xi_1, \ldots, \xi_k$, which satisfy the following commutation relations:

$$\left\{\xi_i,\ \frac{\delta}{\delta\xi_j}\right\} = \delta_{ij} \qquad \left\{\xi_i,\ \xi_j\right\} = \left\{\frac{\delta}{\delta\xi_i},\ \frac{\delta}{\delta\xi_j}\right\} = 0 \qquad i, j = 1, \ldots, k.$$

The algebra $\mathbf{C}(2k)$ is isomorphic to the canonical anticommutation relations (CAR) algebra [BR] over a 2^k-dimensional Hilbert space, since the operators $\delta/\delta\xi_i$ and ξ_i might be interpreted as annihilation and creation operators for a Fermi system with k degrees of freedom.

We end this section by defining kernels of operators acting on $\mathbf{G}(k)$. We recall that to each $L \in \mathbf{C}(2k)$ there corresponds a unique element $\mathrm{Ker}(L)(\xi, \xi')$ of the Grassmann algebra $\mathbf{G}(2k)$, generated by $\xi_1, \ldots, \xi_k$, and ${\xi'}_1, \ldots, {\xi'}_k$, such that

$$(LF)(\xi) = \int_B \mathrm{Ker}(L)(\xi, \xi')\cdot F(\xi')\,\mathcal{D}_k\xi'.$$

3. Evolution on $\mathbf{G}(k)$

Let us consider the evolution given by the equation

$$\frac{\partial f_t}{\partial t} = L\,f_t \tag{3.1}$$

where $f_t \in \mathbf{G}(k)$ and $L \in \mathbf{C}(2k)$.

In this section first we solve equation (3.1) by constructing a kernel for the operator $\exp(tL)$ using standard tools of Grassmannian analysis and writing the solution of (3.1) as an element of $\mathbf{G}(k)$ with coefficients given by certain Berezin integrals, and second, we show that these integrals can be represented as expectations with respect to a properly chosen family of standard Poisson processes.

We introduce some necessary constructions related to the formal description of continuous time evolution on $\mathbf{G}(k)$. In order to do that it is convenient to embed $\mathbf{G}(k)$ into an extended Grassmann algebra $\mathbf{G}_\infty(k)$, and we will proceed in the following way (cf [MIS]). Consider three Grassmann algebras:

the Grassmann algebra $\Gamma_\tau(k)$, $\tau = (t_1, \ldots, t_m)$, generated by

$$\mathbf{1}, \quad \xi_1(t_i), \ldots, \xi_k(t_i); \quad \rho_1(t_i), \ldots, \rho_k(t_i) \qquad i = 1, \ldots, m$$

the Grassmann algebra $\Gamma_{t]}(k)$ generated by

$$\mathbf{1}, \quad \xi_1(s), \ldots, \xi_k(s); \quad \rho_1(s), \ldots, \rho_k(s) \qquad s \leqslant t$$

and the Grassmann algebra $\Gamma_\infty(k)$ generated by

$$\mathbf{1}, \quad \xi_1(s), \ldots, \xi_k(s); \quad \rho_1(s), \ldots, \rho_k(s) \qquad s > 0$$

where

$$\xi_i(s) \cdot \xi_j(t) = -\, \xi_j(t) \cdot \xi_i(s) \qquad \rho_i(s) \cdot \rho_j(t) = -\, \rho_j(t) \cdot \rho_i(s)$$

and

$$\rho_i(s) \cdot \xi_j(t) = -\xi_j(t) \cdot \rho_i(s) \qquad \forall\, i, j, \quad \forall\, s, t > 0.$$

We consider the exterior algebras

$$\mathbf{G}_\tau(k) = \mathbf{G}_0(k) \bigwedge \Gamma_\tau(k)$$

$$\mathbf{G}_{t\,]}(k) = \mathbf{G}_0(k) \bigwedge \Gamma_{t\,]}(k)$$

$$\mathbf{G}_\infty(k) = \mathbf{G}_0(k) \bigwedge \Gamma_\infty(k)$$

where $\mathbf{G}_0(k) = \mathbf{G}(k)$.

Remark. (a) For the construction of Grassmann algebras with an infinite number of generators we also refer the reader to [B1, R, S]. (b) The generators ρ will be used only for 'Fourier transform'-type expressions and play only an auxilliary role here.

Example. The Pauli equation on $\mathbf{G}(1)$.

Let us consider evolution on $\mathbf{G}(1)$, generated by $\mathbf{1}$ and ξ_1, given by (3.1) where

$$L = h_1\,\sigma_1 + h_2\,\sigma_2 + h_3\,\sigma_3$$

and we identify

$$\sigma_1 = \xi + \frac{\delta}{\delta\xi} \qquad \sigma_2 = i\left(\xi - \frac{\delta}{\delta\xi}\right) \qquad \sigma_3 = \frac{1}{i}\sigma_1\sigma_2$$

so that the usual commutation rules of Pauli matrices are satisfied.

As will be proved in theorem 3.1 the kernel of the operator $\exp(tL)$ can be written as the limit

$$\mathrm{Ker}(\mathrm{e}^{tL})(\xi, \xi') = \lim_{m\to\infty} Q_t^m(\xi, \xi')$$

where

$$Q_t^m(\xi,\xi') = \underbrace{\int^B \cdots \int^B}_{m \text{ times}} P_t^m(\xi,\xi_t,\rho_t,\xi')\ \mathcal{D}_m\rho_t\ \mathcal{D}_m\xi_t$$

which is the kernel of the operator $(\mathbf{1}+(t/m)L)^m$ and $P_t^m(\xi,\xi_t,\rho_t,\xi') \in \mathbf{G}_0(1)\bigwedge\Gamma_{\tau(t,m)}(1)$, with $\tau(t,m)=(t/m, 2t/m, \ldots, [(m-1)t]/m, t)$, and has the following form:

$$\begin{aligned}P_t^m(\xi,\xi_t,\rho_t,\xi') = \exp\Bigg\{\sum_{j=0}^{m}\frac{t}{m}\Bigg\{&h_1\left[\xi\left(\frac{j\cdot t}{m}\right)+\rho\left(\frac{j\cdot t}{m}\right)\right]+ih_2\left[\xi\left(\frac{j\cdot t}{m}\right)-\rho\left(\frac{j\cdot t}{m}\right)\right]\\ &+h_3\left[1-2\xi\left(\frac{j\cdot t}{m}\right)\rho\left(\frac{j\cdot t}{m}\right)\right]\\ &-\rho\left(\frac{j\cdot t}{m}\right)\left[\xi\left(\frac{j\cdot t}{m}\right)+\xi\left(\frac{(j-1)t}{m}\right)\right]\Bigg\}\Bigg\}\end{aligned}$$

where

$$\mathcal{D}_m\rho_t = d\rho\left(\frac{t}{m}\right)\cdots d\rho(t)$$

$$\mathcal{D}_m\xi_t = d\xi\left(\frac{t}{m}\right)\cdots d\xi\left(\frac{(m-1)t}{m}\right)$$

and we set $\xi(0)=\xi'$, $\xi(t)=\xi$. In this way the solution of equation (3.1) with initial data $F(\xi)=f_0\cdot\mathbf{1}+f_1\cdot\xi;\ f_0, f_1\in\mathbb{C}$ can be written as

$$F_t(\xi) = (\mathrm{e}^{tL}F)(\xi) = \int^B \left[\mathrm{Ker}(\mathrm{e}^{tL})(\xi,\xi')F(\xi')\right] d\xi' \tag{3.2}$$

and we have

$$F_t(\xi) = f_0(t)\mathbf{1} + f_1(t)\cdot\xi$$

where

$$f_0(t) = -\int^B\int^B \left[\mathrm{Ker}(\mathrm{e}^{tL})(\xi,\xi')F(\xi')\cdot\xi\right] d\xi'\ d\xi \tag{3.3}$$

$$f_1(t) = \int^B\int^B \left[\mathrm{Ker}(\mathrm{e}^{tL})(\xi,\xi')F(\xi')\right] d\xi'\ d\xi. \tag{3.4}$$

On the other hand, from equation (1.2) specialized to the present simplified case, we have that

$$f_0(t) = \mathrm{e}^t\,\mathbf{E}\left[f_{\frac{1}{2}(1-(-1)^{N_t})}\cdot\exp\biggl(\int_0^t \ln(h_1 - i(-1)^{N_\tau}h_2)\ dN_\tau - \int_0^t h_3(-1)^{N_\tau}\ d\tau\biggr)\right] \tag{3.5}$$

and

$$f_1 = \mathrm{e}^t\,\mathbf{E}\left[f_{\frac{1}{2}(1+(-1)^{N_t})}\cdot\exp\biggl(\int_0^t \ln(h_1 + i(-1)^{N_\tau}h_2)\ dN_\tau + \int_0^t h_3(-1)^{N_\tau}\ d\tau\biggr)\right] \tag{3.6}$$

where N_t is a Poisson process with unit parameter. Comparing equations (3.3) and (3.5) we obtain

$$
\begin{aligned}
-\int^B\!\!\int^B \left[\mathrm{Ker}(\mathrm{e}^{tL})(\xi,\xi')F(\xi')\xi\right] d\xi'\, d\xi \\
&= \mathrm{e}^t\, \mathbf{E}\left[f_{\frac{1}{2}(1-(-1)^{N_t})} \exp\biggl(\int_0^t \ln(h_1 - i(-1)^{N_\tau} h_2)\, dN_\tau \right. \\
&\qquad \left. - \int_0^t h_3(-1)^{N_\tau}\, d\tau\biggr)\right]. \qquad (3.7)
\end{aligned}
$$

An analogous equation can be written for $f_1(t)$.

In this way we obtain the equality between certain Berezin integrals and expectations with respect to the standard Poisson process.

To discuss the general case, let us consider the family of operators

$$
\gamma_j = \xi_j + \frac{\delta}{\delta\xi_j} \qquad \overline{\gamma}_j = \frac{1}{i}\left(\xi_j - \frac{\delta}{\delta\xi_k}\right) \qquad j = 1, \ldots, n
$$

which satisfy the commutation relations

$$
\{\gamma_i, \overline{\gamma}_j\} = 0 \qquad \{\gamma_i, \gamma_j\} = \{\overline{\gamma}_i, \overline{\gamma}_j\} = 2\delta_{ij}.
$$

Furthermore, we will use the following notation. Let ν, x, $\mu \in M_k$ where

$$
\nu = \{\nu_1, \ldots, \nu_n\} \qquad x = \{x_1, \ldots, x_\ell\} \qquad \mu = \{\mu_1, \ldots, \mu_m\}
$$

such that

$$
\nu \cap \mu = \nu \cap x = s \cap \mu = \emptyset.
$$

We will write

$$
\gamma^{(\nu,x,\mu)} = \prod_{i=1}^{n} \gamma_{\nu_i} \cdot \prod_{j=1}^{\ell} (\overline{\gamma}_{x_j}\, \gamma_{x_j}) \cdot \prod_{r=1}^{m} \overline{\gamma}_{\mu_r}. \qquad (3.8)
$$

Let us set

$$
L = \sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)}\, \gamma^{(\nu,x,\mu)}
$$

where $\gamma^{(\nu,x,\mu)} \in \mathbf{C}(2k)$ and $h_{(\nu,x,\mu)} \in \mathbb{C}$.

Theorem 3.1. The kernel of the operator $\exp(tL)$ acting on $\mathbf{G}$(k) is given as the limit

$$
\mathrm{Ker}(\mathrm{e}^{tL})(\xi,\xi') = \lim_{m\to\infty} Q_t^m(\xi,\xi') \in \mathbf{G}(2k)
$$

where

$$
\begin{aligned}
Q_t^m(\xi,\xi') = \int^B \exp\Biggl\{ -\sum_{j=1}^{m}\Biggl(-\frac{j\cdot t}{m} \sum_{(\nu,x,\mu)} \Biggl[h_{(\nu,x,\mu)} \prod_{\nu} (\xi_\nu + i\rho_\nu) \\
\cdot \prod_x (2\xi_x\rho_x + i\mathbf{1}) \prod_\mu \frac{1}{i}(\xi_\mu - i\rho_\mu)\Biggr]\Biggr) \\
- \sum_{j}^{m}\sum_{r=1}^{n} \rho_r\left(\frac{j\cdot t}{m}\right)\left[\xi_r'\left(\frac{j\cdot t}{m}\right) + \xi_r'\left(\frac{(j-1)t}{m}\right)\right]\Biggr\}\, \mathcal{D}\rho
\end{aligned}
$$

if k is even; a similar formula holds for k odd (see the appendix).

Proof. The proof is rather simple and we postpone it to the appendix.

In this way we can write the solution of equation (3.1) as the Berezin integral

$$F_t(\xi) = (e^{tL} F_0)(\xi) = \int^B [\mathrm{Ker}(e^{tL})(\xi, \xi') F_0(\xi')] \, \mathcal{D}\xi'$$
$$= \sum_{\mu \in M} f_\mu(t) \cdot \xi^\mu$$

with $f_\mu(t) \in \mathbb{C}$.

Using the Berezin integration rules we immediately obtain

$$f_\mu(t) = \int^B [F_t(\xi) \cdot \xi^{\mu^c}] \, \mathcal{D}\xi = \int^B \left\{ \int^B [\mathrm{Ker}(e^{tL})(\xi, \xi') F_0(\xi')] \, \mathcal{D}\xi' \cdot \xi^{\mu^c} \right\} \mathcal{D}\xi \tag{3.9}$$

where $\mu^c \in M$ is a complementary multi-index to μ, i.e. $\mu = (\mu_1, \dots, \mu_n)$ and $\mu^c = (\mu_1^c, \dots, \mu_{n'}^c)$ are such that $\{\mu_1, \dots, \mu_n\} \bigcap \{\mu_1^c, \dots, \mu_{n'}^c\} = \emptyset$ and $\{\mu_1, \dots, \mu_n\} \bigcup \{\mu_1^c, \dots, \mu_{n'}^c\} = \{1, 2, \dots, k\}$, and we assume here that $\emptyset^c = \{1, 2, \dots, k\}$.

Let us fix a total ordering $\prec$ on M (lexicographic order, for example). It induces a total ordering on the set of monomials ξ^μ in the following way:

$$\xi^\mu \prec \xi^{\mu'} \qquad \text{if } \mu \prec \mu'.$$

Let us rename all monomials with respect to the order $\prec$ by $\xi^1, \xi^2, \dots, \xi^{2^k}$, which form the basis of $\mathbf{G}(k)$ as a 2^k-dimensional linear space.

In this basis the matrix elements of the operator $\gamma^{(\nu,x,\mu)} = \prod_{i=1}^n \gamma_{\nu_i} \cdot \prod_{j=1}^{\ell} (\overline{\gamma}_{x_j} \gamma_{x_j}) \cdot \prod_{r=1}^m \overline{\gamma}_{\mu_r}$ can be computed explicitly, so equation (3.1) could be rewritten as

$$\frac{\partial f_\alpha(t)}{\partial t} = \sum_{\beta=1}^{2^k} L_{\alpha,\beta} \, f_\beta(t) \tag{3.10}$$

where the coefficients $L_{\alpha,\beta} \equiv L_{\alpha,\beta}(h_1, h_2, \dots, h_{2^k}) \in \mathbb{C}$.

With an obvious definition of $\Phi(\alpha, \beta)$ and $\Psi(\alpha)$, equation (3.10) can be rewritten as

$$\frac{\partial f_\alpha(t)}{\partial t} = \sum_{\beta=1}^{2^k - 1} \exp[\, \Phi(\alpha, \beta)] \cdot f_{\alpha \oplus \beta}(t) + \Psi(\alpha) \cdot f_\alpha(t) \tag{3.11}$$

with the initial condition

$$f_\alpha(0) = f_\alpha$$

and where the sign $\oplus$ stands for the sum modulo 2^k.

The solution of the linear system (3.11) is given by

$$f_\alpha(t) = e^{(2^k - 1) \cdot t} \cdot \mathbf{E}\Bigg[f_{\alpha \oplus N_t} \cdot \exp\Bigg(\int_0^t \Psi(\alpha \oplus N_t \ominus N_\tau) \, d\tau$$
$$+ \sum_{\beta=1}^{2^k - 1} \int_0^t \Phi(\alpha \oplus N_t \ominus N_\tau \ominus \beta, \beta) \, dN_\tau^\beta \Bigg) \Bigg] \tag{3.12}$$

where $N_t = \sum_{\beta=1}^{2^k-1} \beta N_t^\beta$ is the sum of $2^k - 1$ independent Poisson processes and $\alpha \oplus N_t$, $\alpha \oplus N_t \ominus N_\tau$, and $\alpha \oplus N_t \ominus N_\tau \ominus \beta$ stand for sums and differences modulo 2^k. (For a complete proof see [DJLS].)

Now comparing equations (3.9) and (3.12)

$$\int^B \left\{ \int^B [\mathrm{Ker}(\mathrm{e}^{tL})(\xi, \xi') F_0(\xi')] \, \mathcal{D}_k\xi' \cdot \xi^{\mu^c} \right\} \mathcal{D}_k\xi$$

$$= \mathrm{e}^{(2^k-1)t} \cdot \mathbf{E}\Bigg[f_{\alpha\oplus N_t} \cdot \exp\Bigg(\int_0^t \Psi(\alpha \oplus N_t \ominus N_\tau) \, d\tau + \sum_{\beta=1}^{2^k-1} \int_0^t \Phi(\alpha \oplus N_t \ominus N_\tau \ominus \beta, \ \beta) \, dN_\tau^\beta \Bigg) \Bigg] \tag{3.13}$$

where α corresponds to μ in the ordering. Sums and differences modulo 2^k are slightly unconfortable to handle. In the next three sections we shall reformulate the theory in a space of functions of dichotomic variables. This allows the construction of a systematic formalism suitable for both theoretical and numerical analysis.

4. Representation of Grassmann algebras in the space of functions of dichotomic variables (σ-representation)

In this section we discuss a linear bijection between Grassmann algebras and spaces of functions of dichotomic variables.

Let $\mathbf{Z}_2$ be $\{-1, 1\}$ with its natural Abelian group structure and $\mathbf{Z}_2^{\times k}$ be the direct product of k copies of $\mathbf{Z}_2$ which is a finite commutative group with the unit element $e_k = (1, \ldots, 1)$.

Let us define $\mathcal{H}_k$ as the linear space of all 'wavefunctions' $\chi(\cdot) : \mathbf{Z}_2^{\times k} \to \mathbb{C}$ of k dichotomic variables $\sigma_1, \ldots, \sigma_k$. It becomes a 2^k-dimensional Hilbert space when equipped with the inner product

$$\langle \chi_1 \, , \chi_2 \rangle_k = \sum_{\sigma \in \mathbf{Z}_2^{\times k}} \overline{\chi}_1(\sigma) \chi_2(\sigma) \tag{4.1}$$

and could be interpreted as the space of (pure) states for a Heisenberg ferromagnet in a finite box.

All monomials of the Grassmann algebra $\mathbf{G}(k)$ can be indexed by elements of $\mathbf{Z}_2^{\times k}$ in the following way:

$$\sigma \mapsto \xi(\sigma) \equiv \xi^{\frac{1}{2}(1-\sigma)} = \xi_1^{\frac{1}{2}(1-\sigma_1)} \cdots \xi_k^{\frac{1}{2}(1-\sigma_k)} \qquad \sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbf{Z}_2^{\times k} \tag{4.2}$$

and let us define a map $\mathcal{I}$: $\mathcal{H}_k \to \mathbf{G}(k)$ by the formula

$$\mathcal{I}(\chi) \equiv F_\chi(\xi) = \sum_{\sigma \in \mathbf{Z}_2^{\times k}} \chi(\sigma) \cdot \xi^{\frac{1}{2}(1-\sigma)} \in \mathbf{G}(k) \tag{4.3}$$

where the 'wavefunction' $\chi(\cdot) \in \mathcal{H}_k$.

It is easy to see that the map $\mathcal{I}$ is injective, i.e. $\mathcal{I}(\chi_1) \neq \mathcal{I}(\chi_2)$ if $\chi_1 \neq \chi_2$. Moreover, the map $\mathcal{J} : \mathbf{G}(k) \to \mathcal{H}_k$ defined by

$$\mathcal{J}(F(\xi)) \equiv \chi_F(\sigma) = \triangle_k(\sigma) \int^B \xi^{\frac{1}{2}(1+\sigma)} \cdot F(\xi) \, \mathcal{D}_k\xi \tag{4.4}$$

where $\triangle_k(\cdot) : \mathbf{Z}_2^{\times k} \to \mathbf{Z}_2$ is given by

$$\triangle_k(\sigma) = \prod_{l=1}^{k} \left(\frac{1-\sigma_l}{2} + \frac{1+\sigma_l}{2} \cdot \sigma_1 \cdots \sigma_{l-1} \right) \tag{4.5}$$

is the inverse of $\mathcal{I}$: $\mathcal{I} = \mathcal{J}^{-1}$, and, clearly, it is surjective. By this we obtain a linear bijection between Grassmann algebra $\mathbf{G}(k)$ and the space of functions of dichotomic variables $\mathcal{H}_k$, which we call the σ-representation.

Next we turn to $\mathbf{C}(2k)$. To each $\widehat{A} \in \mathbf{C}(2k)$ there corresponds a linear operator $A : \mathcal{H}_k \to \mathcal{H}_k$ given by the formula

$$\widehat{A}\, F(\xi) = \sum_{\sigma \in \mathbf{Z}_2^{\times k}} (A\chi)(\sigma) \cdot \xi^{\frac{1}{2}(1-\sigma)}. \tag{4.6}$$

For any $\widehat{A} \in \mathbf{C}(2k)$, which is a linear combination of normally ordered products $\xi^{\frac{1}{2}(1-\epsilon)} \cdot (\delta/\delta\xi)^{\frac{1}{2}(1-\eta)}$, where $\epsilon, \eta \in \mathbf{Z}_2^{\times k}$, we find its image A by computing the image of the operators $\xi^{\frac{1}{2}(1-\epsilon)} \cdot (\delta/\delta\xi)^{\frac{1}{2}(1-\eta)}$ which we denote by $a^{*\frac{1}{2}(1-\epsilon)} a^{\frac{1}{2}(1-\eta)}$.

Proposition 4.1. For all $\epsilon, \eta \in \mathbf{Z}_2^{\times k}$ we have

$$(a^{*\frac{1}{2}(1-\epsilon)}\chi)(\sigma) = C_k(\epsilon, \sigma) \cdot \chi(\epsilon\sigma)$$

$$(a^{\frac{1}{2}(1-\eta)}\chi)(\sigma) = A_k(\eta, \sigma) \cdot \chi(\eta\sigma)$$

$$(a^{*\frac{1}{2}(1-\epsilon)} a^{\frac{1}{2}(1-\eta)}\chi)(\sigma) = N_k(\epsilon, \eta, \sigma) \cdot \chi(\epsilon\eta\sigma)$$

where

$$C_k(\epsilon, \sigma) = \prod_{l=1}^{k} \left(\frac{1+\epsilon_l}{2} + \frac{1-\epsilon_l}{2} \cdot \frac{1-\sigma_l}{2} \cdot \epsilon_1 \cdots \epsilon_{l-1} \cdot \sigma_1 \cdots \sigma_{l-1} \right)$$

$$A_k(\eta, \sigma) = \prod_{l=1}^{k} \left(\frac{1+\eta_l}{2} + \frac{1-\eta_l}{2} \cdot \frac{1+\sigma_l}{2} \cdot \eta_1 \cdots \eta_{l-1} \cdot \sigma_1 \cdots \sigma_{l-1} \right)$$

$$N_k(\epsilon, \eta, \sigma) = A_k(\eta, \sigma) \cdot C_k(\epsilon, \eta\sigma)$$

and $\epsilon\sigma = \epsilon_1\sigma_1, \ldots, \epsilon_k\sigma_k$.

Proof. This immediately follows from the fact that

$$\xi^{\frac{1}{2}(1-\epsilon)} \xi^{\frac{1}{2}(1-\sigma)} = C(\epsilon, \epsilon\sigma) \cdot \xi^{\frac{1}{2}(1-\epsilon\sigma)} \qquad \left(\frac{\delta}{\delta\xi}\right)^{\frac{1}{2}(1-\eta)} \xi^{\frac{1}{2}(1-\sigma)} = A(\eta, \eta\sigma) \cdot \xi^{\frac{1}{2}(1-\eta\sigma)}$$

which can be checked by induction on k.

Corollary. If $F(\xi) = \sum_{\epsilon \in \mathbf{Z}_2^{\times k}} f(\epsilon)\xi^{\frac{1}{2}(1-\epsilon)} \in \mathbf{G}(k)$, the image $F(a^*)$ of the linear operator $F(\xi) \in \mathbf{C}(2k)$ is

$$(F(a^*)\chi)(\sigma) = \sum_{\epsilon \in \mathbf{Z}_2^{\times k}} f(\epsilon) C_k(\epsilon, \sigma)\chi(\epsilon\sigma) \tag{4.7}$$

and, in particular, when $F(\xi) = \sum_{1 \leqslant i < j \leqslant 2k} A_{ij}\xi_i\xi_j$

$$\sum_{1 \leqslant i < j \leqslant 2k} A_{ij}(a_i^* a_j^* \chi)(\sigma) = \sum_{(i,j) \in \Gamma(A)} A_{ij} \frac{1-\sigma_i}{2} \frac{1-\sigma_j}{2}$$

$$\times \prod_{l=i+1}^{j-1} \sigma_l \chi(\sigma_1, \ldots, -\sigma_i, \ldots, -\sigma_j, \ldots, \sigma_{2k}) \tag{4.8}$$

where $\Gamma(A) = \{(i, j), 1 \leqslant i < j \leqslant 2k : A_{ij} \neq 0\}$.

Let A be a linear operator acting on $\mathcal{H}_k$ and $A(\cdot,\cdot)$ its matrix, given by

$$(A\chi)(\sigma) = \sum_{\sigma'\in\mathbf{Z}_2^{\times k}} A(\sigma,\sigma')\chi(\sigma'). \tag{4.9}$$

For instance, from proposition 4.1 we obtain that the matrix element of the operator $a^{*\frac{1}{2}(1-\epsilon)}a^{\frac{1}{2}(1-\eta)}$ is given by

$$A(\sigma,\sigma') = N_k(\epsilon,\eta,\sigma)\delta_{\epsilon\sigma,\eta\sigma'}.$$

Now from equation (4.4) we obtain

$$\int^B \xi^{\frac{1}{2}(1+\sigma)} A\, \xi^{\frac{1}{2}(1-\sigma')}\, \mathcal{D}_k\xi = \triangle_k(\sigma)A(\sigma,\sigma') \qquad \forall A\ \in \mathbf{C}(2k) \tag{E.1}$$

which relates the Berezin integral $\int^B \xi^{\frac{1}{2}(1+\sigma)}\hat{A}\xi^{\frac{1}{2}(1-\sigma')}\,\mathcal{D}_k\xi$ to the matrix $A(\sigma,\sigma')$. This is our first basic formula.

We end this section by some remarks about the kernels of $A\in\mathbf{C}(2k)$. We recall that to each $A\in\mathbf{C}(2k)$ there corresponds a unique $\mathrm{Ker}(A)(\xi,\xi')\in\mathbf{G}^2(k)$, the kernel of the linear operator A, such that

$$AF(\xi) = \int^B \mathrm{Ker}(A)(\xi,\xi')F(\xi')\,\mathcal{D}_k\xi' \qquad \forall\, F(\xi)\in\mathbf{G}(k). \tag{4.11}$$

It can easily be seen that for all positive integer k the kernel $\mathrm{Ker}(\mathbf{1}_{2k})(\xi,\xi')$ of the unit element $\mathbf{1}_{2K}$ of $\mathbf{C}(2k)$ is

$$\mathrm{Ker}(\mathbf{1}_{2k})(\xi,\xi') = \sum_{\sigma\in\mathbf{Z}_2^{\times k}} \triangle_k(\sigma)\xi^{\frac{1}{2}(1-\sigma)}\xi'^{\frac{1}{2}(1+\sigma)} \tag{4.12}$$

and more generally

$$\mathrm{Ker}(A)(\xi,\xi') = \sum_{\sigma,\sigma'\in\mathbf{Z}_2^{\times k}} \triangle_k(\sigma')A(\sigma,\sigma')\xi^{\frac{1}{2}(1-\sigma)}\xi'^{\frac{1}{2}(1+\sigma')} \tag{4.13}$$

which, together with

$$A(\sigma,\sigma') = \triangle_k(\sigma)\iint^B \xi^{\frac{1}{2}(1+\sigma)}\,\mathrm{Ker}(A)(\xi,\xi')\xi'^{\frac{1}{2}(1-\sigma)}\; d\xi'_k\cdots d\xi'_1\; d\xi_k\cdots d\xi_1 \tag{4.14}$$

relates the kernel $\mathrm{Ker}(A)(\xi,\xi')$ of A and the matrix $A(\sigma,\sigma')$.

5. Semigroups associated with Poisson processes

We now turn to semigroups of linear operators acting on the Hilbert space $\mathcal{H}_k$. We shall give a probabilistic representation of the semigroup $\{\exp tL\}_{t\geqslant 0}$ generated by a non-trivial linear operator $L:\mathcal{H}_k\to\mathcal{H}_k$. It specializes the more general formulae introduced in [DJLS] to which we refer the reader; nevertheless this section will be self-contained.

We start with a special representation of $L\in L(\mathcal{H}_k,\mathcal{H}_k)$. For all $\epsilon=(\epsilon_1,\ldots,\epsilon_k)\in\mathbf{Z}_2^{\times k}$, and different from the identity $\epsilon\neq e_k=(1,\ldots,1)$, let D^ϵ be the self-adjoint difference operator

$$(D^\epsilon\chi)(\sigma) = \chi(\epsilon\sigma) - \chi(\sigma) \tag{5.1}$$

which annihilates constants.

We observe that each $L\in L(\mathcal{H}_k,\ \mathcal{H}_k)$ admits the representation

$$L = \sum_{\epsilon\neq\epsilon_k} \lambda_\epsilon(\cdot)D^\epsilon\ -\ V(\cdot)\mathbf{1}$$

where the functions $\lambda_\epsilon(\cdot), V(\cdot) : \mathbf{Z}_2^{\times k} \to \mathbb{C}$ are related to the matrix $L(\cdot,\cdot)$ of the operator L by the formulae

(i) $\qquad \lambda_\epsilon(\sigma) = L(\sigma, \epsilon\sigma)$

(ii) $\qquad V(\sigma) = -\sum_{\sigma' \in \mathbf{Z}_2^{\times k}} L(\sigma, \sigma').$

Let $\Gamma(L)$ be the collection of $\epsilon \neq e_k$ such that $\lambda_\epsilon(\cdot)$ is not identically vanishing and $|\Gamma(L)|$ be its cardinality. If $\Gamma(L) \neq \emptyset$ we call the operator L a difference operator.

Now let $\{N_t^\epsilon\}_{\epsilon \neq e_k}$ be a given collection of $(2^k - 1)$ independent Poisson processes of unit parameter which we assume to be left-continuous.

Theorem 5.1 (probabilistic representation of semigroups). Let $L = \sum_{\epsilon\in\Gamma(L)} \lambda_\epsilon(\cdot) D^\epsilon - V(\cdot)\mathbf{1}$ be a difference operator. Then

$$(\mathrm{e}^{tL}\chi)(\sigma) = \mathrm{e}^{t|\Gamma(L)|}\mathbf{E}\Bigg(\chi(\sigma(-1)^{N_t}) \exp\Bigg\{\sum_{\epsilon\in\Gamma(L)}\Bigg(\int_{[0,t)} \ln\lambda_\epsilon(\sigma(-1)^{N_s})\, dN_s^\epsilon \\ - \int_0^t \lambda_\epsilon(\sigma(-1)^{N_s})\, ds\Bigg) - \int_0^t V(\sigma(-1)^{N_s})\, ds\Bigg\}\Bigg)$$

where $N_s = (N_s^1, \dots, N_s^k)$ with $N_s^l = \sum_{\epsilon\in\Gamma(L)} \frac{1}{2}(1-\epsilon_l) N_s^\epsilon$.

Remark. By convention $\exp\int_{[0,t)} \ln\lambda_\epsilon(\sigma(-1)^{N_s})\, dN_s^\epsilon$ vanishes if $\lambda_\epsilon(\sigma(-1)^{N_s}) = 0$ for some $0 < s < t$ such that $N_{s+}^\epsilon - N_s^\epsilon \neq 0$. We observe that it does not depend upon the choice of the branch of the logarithm.

Proof. We follow the strategy explained in [DJLS].
(i) The right-hand side defines a semigroup $\{P^t\}_{t\geqslant 0}$ of linear operators on $\mathcal{H}_K$ by the Markov property of Poisson processes. In order to complete the proof we must show that the infinitesimal generator of $\{P^t\}_{t\geqslant 0}$ coincides with L.
(ii) Since each wavefunction $\chi(\cdot)$ is a linear superposition of characters $\chi_n(\sigma) = \sigma^n = \sigma_1^{n_1}\cdots\sigma_k^{n_k}$, $n = (n_1, \dots, n_k) \in (\mathbf{Z}_2^*)^{\times k}$, $\mathbf{Z}_2^* \equiv \{0, 1\}$, we can only consider the case of $\chi(\cdot) = \chi_n(\cdot)$ for some $n \in (\mathbf{Z}_2^*)^{\times k}$. Let $\xi_t^{\sigma,n}$ be the random variable

$$\xi_t^{\sigma,n} = \sum_{\epsilon\in\Gamma(L)} \int_{[0,t)} b_{\epsilon,n}(\sigma(-1)^{N_s})\, dN_s^\epsilon + \int_0^t m(\sigma(-1)^{N_s})\, ds$$

where

$$b_{\epsilon,n}(\sigma) = i\pi\sum_{l=1}^k \frac{1-\epsilon_l}{2} n_l + \ln\lambda_\epsilon(\sigma)$$

$$m(\sigma) = -\sum_{\epsilon\in\Gamma(L)} \lambda_\epsilon(\sigma) - V(\sigma).$$

By definition

$$(P^t\chi_n)(\sigma) = \chi_n(\sigma)\mathrm{e}^{t|\Gamma(L)|}\mathbf{E}(\exp\xi_t^{\sigma,n}).$$

(iii) The stochastic differential $d\exp\xi_t^{\sigma,n}$ of the process $t \in [0, +\infty) \to \exp\xi_t^{\sigma,n}$ can be explicitly evaluated as explained in [DJLS]. It turns out that

$$d\exp\xi_t^{\sigma,n} = (\exp\xi_t^{\sigma,n})\Bigg[m(\sigma(-1)^{N_t})\, dt + \sum_{\epsilon\in\Gamma(L)}\left(\exp\left(b_{\epsilon,n}(\sigma(-1)^{N_t})\right) - 1\right) dN_t^\epsilon\Bigg]$$

$$= (\exp \xi_t^{\sigma,n}) \Bigg[-V(\sigma(-1)^{N_t})\, dt + \sum_{\epsilon \in \Gamma(L)} \lambda_\epsilon(\sigma(-1)^{N_t})(\epsilon^n\, dN_t^\epsilon - dt)$$

$$- \sum_{\epsilon \in \Gamma(L)} dN_t^\epsilon \Bigg].$$

By taking the expectation of $d \exp \xi_t^{\sigma,n}$ for $t = 0$, since $\mathbf{E}(dN_t^\epsilon) = dt$ and $(D^\epsilon \chi_n)(\sigma) = (\epsilon^n - 1)\chi_n)(\sigma)$, it follows that $\big(d(P^t \chi_n)(\sigma)/dt\big)\big|_{t=0} = (L\chi_n)(\sigma)$. Therefore the infinitesimal generator of the semigroup $\{P^t\}_{t \geqslant 0}$ is exactly the difference operator L.

Example. As an elementary illustration of theorem 5.1, let $k = 1$ and $(D\chi)(\sigma) = \chi(-\sigma) - \chi(\sigma)$; then

$$(\mathrm{e}^{tD}\chi)(\sigma) = \mathbf{E}(\chi(\sigma(-1)^{N_t}))$$

where N_t is the Poisson process with unit parameter. Indeed

$$\mathbf{E}(\chi(\sigma(-1)^{N_t})) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-t} t^n}{n!} \chi(\sigma(-1)^n)$$

$$= \mathrm{e}^{-t}\big\{\chi(\sigma)\cosh t + \chi(-\sigma)\sinh t\big\} = (\mathrm{e}^{tD}\chi)(\sigma).$$

From theorem 5.1 we obtain the matrix elements of the operator e^{tL}:

$$\mathrm{e}^{tL}(\sigma, \sigma') = \mathrm{e}^{t|\Gamma(L)|} \mathbf{E}\Bigg(\prod_{l=1}^{k} \frac{1 + \sigma_l \sigma'_l (-1)^{N_t^l}}{2} \exp\Bigg\{ \sum_{\epsilon \in \Gamma(L)} \Bigg(\int_{[0,t)} \ln \lambda_\epsilon(\sigma(-1)^{N_s})\, dN_s^\epsilon$$

$$- \int_0^t \lambda_\epsilon(\sigma(-1)^{N_s})\, ds \Bigg) - \int_0^t V(\sigma(-1)^{N_s})\, ds \Bigg\} \Bigg) \tag{E.2}$$

which is the second important equality, which will be used in section 6 to evaluate Berezin integrals.

6. Berezin integrals and Poisson processes in the σ-representation

We shall consider Berezin integrals of the form

$$\int^B \xi^n \exp(-S(\xi))\, \mathcal{D}_k \xi = \int^B \xi_1^{n_1} \xi_2^{n_2} \cdots \xi_k^{n_k} \exp(-S(\xi))\, \mathcal{D}_k \xi \tag{6.1}$$

where $n = (n_1, \ldots, n_k) \in (\mathbf{Z}_2^*)^{\times k}$ and $S(\xi) \neq c \cdot \mathbf{1}_k$ is a non-trivial element of $\mathbf{G}(k)$. It could be interpreted as the 'action' for the (Euclidean) 'Fermi field' $\{\xi_1, \ldots, \xi_k\}$ in which case (6.1) would provide all (unnormalized) 'correlation functions' or (Euclidean) 'Green functions' of the field. In particular, for $n = (0,\ 0, \ldots, 0)$, equation (6.1) gives the 'partition function':

$$Z[S] = \int^B \exp(-S(\xi))\, \mathcal{D}_k \xi. \tag{6.2}$$

Remark. We could easily consider more general integrals by taking for S an element of $\mathbf{C}$(2k). Integrals of the form (6.1) cover a large class of physical applications.

Let $s(\cdot) : \mathbf{Z}_2^{\times k} \to \mathbb{C}$ be defined as

$$s(\epsilon) = \triangle_k(\epsilon) \int^B \xi^{\frac{1}{2}(1+\epsilon)} S(\xi)\, \mathcal{D}_k \xi \tag{6.3}$$

with $\triangle_k(\epsilon)$ as in (4.5).

Theorem 6.1 (Berezin integrals as Poisson averages). For each $k = 1, 2, \ldots,$ each non-trivial $S(\xi) \in \mathbf{G}_k$ and for all $n = (n_1, \ldots, n_k) \in (\mathbf{Z}_2^*)^{\times k}$

$$\int^B \xi_1^{n_1} \cdots \xi_k^{n_k} \exp(-S(\xi))\, \mathcal{D}_k \xi$$

$$= \triangle_k(-(-1)^n) \exp\big((|\Gamma(S)| - s(e_k))\big)\mathbf{E}\Bigg(\prod_{l=1}^{k} \frac{1-(-1)^{N_1^l+n_l}}{2}$$

$$\times \exp\left\{\sum_{\epsilon\in\Gamma(S)} \int_{[0,1)} \ln\big(-s(\epsilon)C_k\big(\epsilon, -(-1)^{N_s+n}\big)\big)\ dN_s^\epsilon\right\}\Bigg) \tag{E.3}$$

where $s(\epsilon)$ is given by (6.3).

In particular

$$Z[S] = \mathrm{e}^{(|\Gamma(S)|-s(e_k))}\mathbf{E}\Bigg(\prod_{l=1}^{k} \frac{1-(-1)^{N_1^l}}{2} \exp\left\{\sum_{\epsilon\in\Gamma(S)} \int_{[0,1)} \ln\big(-s(\epsilon)C_k\big(\epsilon, -(-1)^{N_s}\big)\big)\ dN_s^\epsilon\right\}\Bigg). \tag{6.5}$$

Proof. Using equation (4.4) we obtain

$$S(\xi) = \sum_{\epsilon\in\mathbf{Z}_2^{\times k}} s(\epsilon)\xi^{\frac{1}{2}(1-\epsilon)} \tag{6.6}$$

and, by hypothesis, the subset $\Gamma(S) = \{\epsilon \in \mathbf{Z}_2^{\times k}, \epsilon \neq e_k : s(\epsilon) \neq 0\}$ is non empty. To the operator $-S(\hat{\xi}) \in \mathbf{C}(2k)$ there corresponds the image $L = -S(a^*) \in L(\mathcal{H}_k, \mathcal{H}_k)$, which is the difference operator

$$L = \sum_{\epsilon\in\Gamma(S)} \lambda_\epsilon(\cdot)D^\epsilon - V(\cdot)\mathbf{1} \tag{6.7}$$

with

$$\lambda_\epsilon(\sigma) = -s(\epsilon)C_k(\epsilon,\sigma) \tag{6.8}$$

$$V(\sigma) = \sum_{\epsilon\in\mathbf{Z}_2^{\times k}} s(\epsilon)C_k(\epsilon,\sigma) = s(e_k) - \sum_{\epsilon\in\Gamma(S)} \lambda_\epsilon(\sigma). \tag{6.9}$$

From equations (E.1), (E.2) and the equality

$$-V(\sigma) - \sum_{\epsilon\in\Gamma(S)} \lambda_\epsilon(\sigma) = -s(e_k)$$

it follows that

$$-\int_0^1 V(-(-1)^{N_s+n})\, ds - \sum_{\epsilon\in\Gamma(S)} \int_0^1 \lambda_\epsilon(-(-1)^{N_s+n})\, ds = -s(e_k).$$

In the above $C_k(\epsilon,\sigma)$ is as defined in proposition 4.1 and N_s, N_s^l are as defined in theorem 5.1.

We now make an important remark.

Remark. In equation (E.3) only the trajectories of the Poisson processes with zero or one jump contribute to the expectation. In fact, as soon as one of the factors $(1 + (-1)^{N_t^r+n_r})$ in C_k vanishes, the stochastic integral equals $-\infty$. On the other hand, N_t^r is a sum of independent processes and the event of two processes jumping at the same instant has zero probability.

We now take advantage of the fact that the calculation of a Berezin integral has been reduced to an ordinary integral and we derive simple estimates. From theorem 6.1 we obtain

$$\begin{aligned}
&\left|\int^{B}\xi_1^{n_1}\cdots\xi_k^{n_k}\exp(-S(\xi))\,\mathcal{D}_k\xi\right| \\
&\qquad\leqslant \exp\big((|\Gamma(S)|-\operatorname{Re}s(e_k))\big)\,\mathbf{E}\Bigg(\prod_{l=1}^{k}\frac{1-(-1)^{N_1^l+n_l}}{2} \\
&\qquad\quad\times\exp\Bigg\{\sum_{\epsilon\in\Gamma(S)}\int_{[0,1)}\ln\big(|s(\epsilon)|\big|C_k(\epsilon,-(-1)^{N_s+n})\big|\big)\,dN_s^{\epsilon}\Bigg\}\Bigg) \\
&\qquad\leqslant \exp\big((|\Gamma(S)|-\operatorname{Re}s(e_k))\big)\,\mathbf{E}\Bigg(\chi_{\{N_1^{\epsilon}=0,1\}}\prod_{l=1}^{k}\frac{1-(-1)^{N_1^l+n_l}}{2}\prod_{\epsilon\in\Gamma(S)}|s(\epsilon)|^{N_1^{\epsilon}}\Bigg) \\
&\qquad= \mathrm{e}^{(-\operatorname{Re}s(e_k))}\,\frac{1}{2^k}\sum_{\rho_1,\ldots,\rho_k=0,1}(-1)^{\sum_{i=1}^{k}\rho_i(1+n_i)} \\
&\qquad\quad\times\prod_{\epsilon\in\Gamma(S)}\Big\{1+(-1)^{\sum_{i=1}^{k}\rho_i\frac{1}{2}(1-\epsilon_i)}|s(\epsilon)|\Big\}
\end{aligned}\tag{6.10}$$

since $|C_k(\epsilon,\sigma)|\leqslant 1$ and $\int_{[0,1)}dN_s^{\epsilon}=N_1^{\epsilon}$; χ is the characteristic function of the event indicated.

7. Gaussian Berezin integrals

Gaussian Berezin integrals are a particular case when the 'action' S is bilinear in the 'Fermi field'. Let $\mathbf{G}^2(k)$ be the Grassmann algebra generated by $\{\overline{\eta}_1,\ldots,\overline{\eta}_k;\eta_1,\ldots,\eta_k\}$ and $(B)_{ij}$ any $k\times k$ matrix. Let us consider the integral

$$\int^{B}\overline{\eta}_1^{\overline{\nu}_1}\cdots\overline{\eta}_k^{\overline{\nu}_k}\eta_1^{\nu_1}\cdots\eta_k^{\nu_k}\exp\Bigg\{-\frac{1}{2}\sum_{i,j=1}^{k}B_{ij}\overline{\eta}_i\eta_j\Bigg\}\,\mathcal{D}_k\eta\,\mathcal{D}_k\overline{\eta}\tag{7.1}$$

where $\overline{\nu}_l,\nu_l\in\mathbf{Z}_2^*=\{0,1\}$. Making the substitution $\xi_1=\overline{\eta}_1,\ldots,\xi_k=\overline{\eta}_k;\ \xi_{k+1}=\eta_1,\ldots,\xi_{2k}=\eta_k$, we obtain

$$\sum_{i,j=1}^{k}B_{ij}\overline{\eta}_i\eta_j=\sum_{r,s=1}^{2k}A_{rs}\xi_r\xi_s$$

where (A_{rs}) is the $2k\times 2k$ antisymmetric matrix defined by

$$A_{rs}=\begin{cases}2^{-1}B_{r(s-k)} & \text{when } r=1,\ldots,k \text{ and } s=k+1,\ldots,2k\\ -2^{-1}B_{s(r-k)} & \text{when } r=k+1,\ldots,2k \text{ and } s=1,\ldots,k\\ 0 & \text{otherwise.}\end{cases}\tag{7.2}$$

Therefore the Gaussian integral (7.1) can be written in the following form:

$$\begin{aligned}
&\int^{B}\overline{\eta}_1^{\overline{\nu}_1}\cdots\overline{\eta}_k^{\overline{\nu}_k}\eta_1^{\nu_1}\cdots\eta_k^{\nu_k}\exp\Bigg\{-\frac{1}{2}\sum_{i,j=1}^{k}B_{ij}\overline{\eta}_i\eta_j\Bigg\}\,\mathcal{D}_k\eta\,\mathcal{D}_k\overline{\eta} \\
&\qquad=\int^{B}\xi_1^{n_1}\cdots\xi_{2k}^{n_k}\exp\Bigg(-\frac{1}{2}\sum_{r,s=1}^{2k}A_{rs}\xi_r\xi_s\Bigg)\,\mathcal{D}_{2k}\xi
\end{aligned}\tag{7.3}$$

and from now on we use the last form. Let $\Gamma(A)$ be the set:

$$\Gamma(A) = \{(r,s),\ 1 \leqslant r < s \leqslant 2k : A_{rs} \neq 0\} \tag{7.4}$$

which we suppose is not empty. From equation (4.8) we obtain the result that the generator L of the corresponding semigroup is given by

$$(L\psi)(\sigma) = \sum_{(r,s)\in\Gamma(A)} A_{rs} \frac{1-\sigma_r}{2} \frac{1-\sigma_s}{2} \prod_{l=r}^{s-1} \sigma_l \psi(\sigma_1, \ldots, -\sigma_r, \ldots, -\sigma_s, \ldots, \sigma_{2k}) \tag{7.5}$$

and therefore

$$\int^B \xi_1^{n_1} \cdots \xi_{2k}^{n_{2k}} \exp\left(-\frac{1}{2} \sum_{r,s=1}^{2k} A_{rs}\xi_r\xi_s \right) \mathcal{D}_{2k}\xi$$

$$= \triangle_k(-(-1)^n)\mathrm{e}^{|\Gamma(A)|}\mathbf{E}\Bigg(\prod_{l=1}^{2k} \frac{1-(-1)^{N_1^l+n_l}}{2} \prod_{(r,s)\in\Gamma(A)} (A_{rs})^{N_1^{(r,s)}}$$

$$\times \exp\Bigg\{ \sum_{(r,s)\in\Gamma(A)} \int_{[0,1)} \ln\Bigg[\frac{(1+(-1)^{N_t^r+n_r})(1+(-1)^{N_t^s+n_s})}{4}$$

$$\times \prod_{l=r}^{s-1} (-1)^{s-r}(-1)^{N_t^l+n_l} \Bigg] dN_t^{(r,s)} \Bigg\}\Bigg) \tag{E.4}$$

where $\{N_t^{(r,s)}\}_{(r,s)\in\Gamma(A)}$ is a family of independent Poisson processes with unit parameter and

$$N_t^l = \sum_{(l,m)\in\Gamma(A)} N_t^{(l,m)} + \sum_{(m,l)\in\Gamma(A)} N_t^{(m,l)} \qquad \text{for all } 1 \leqslant l \leqslant 2k.$$

Example. Let $k = 1$, $n = (0,0)$ and $(A_{rs}) = i\sigma_2$ (Pauli matrix), then

$$-1 = \int^B \mathrm{e}^{-\bar{\eta}\eta}\, d\eta\, d\bar{\eta} = \int^B \exp\left(-\frac{1}{2}\Sigma_{r,s=1}^2 A_{rs}\xi_r\xi_s \right) \mathcal{D}_2\xi$$

$$= \mathrm{e}\mathbf{E}\left(\frac{1-(-1)^{N_1}}{2} \exp\int_{[0,1)} \ln\left[-(-1)^{N_t} \frac{1+(-1)^{N_t}}{2} \right] dN_t \right). \tag{7.7}$$

This equality can be checked directly: indeed the random variable under expectation vanishes unless the Poisson process has exactly one jump for some $0 < s < 1$, in which case $N_1 = 1$, and which happens with probability e^{-1}.

Remark. Suppose we want to use our representation of Gaussian Berezin integrals in a numerical simulation. Let $2k$ be the total number of η and $\bar{\eta}$. The generic case will require the generation of $k(2k-1)$ independent Poisson processes. For a Dirac field on a finite d-dimensional lattice Λ, the maximum number of Poisson processes involved is $L|\Lambda|(2L|\Lambda|-1))$ where L is the number of components of the field. This is a large number. However, if we want to calculate the propagator of the free field, for example, we need far fewer. In fact, the matrix B_{ij} which is the lattice version of the differential operator $\partial\!\!\!/ + A\!\!\!/ - M$ couples only nearest neighbours. Therefore the required number of independent Poisson processes is reduced to roughly $dL|\Lambda|$, since each site has $2d$ nearest neighbours.

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Note added in proof. After the completion of this paper we became aware that an article by Brydges and Munoz Maya (Brydges D and Munoz Maya I 1991 An application of Berezin integration to large deviations *J. Theor. Prob.* **4** 371) contains, in a different context, some related ideas.

Appendix

Proposition A.1. The kernel of the operator $\gamma^{(\nu,x,\mu)}$ given by (3.8) and acting on $\mathbf{G}(k)$ is given by the following formulae:

$$\operatorname{Ker}\gamma^{(\nu,x,\mu)}(\xi,\xi') = \int^B \prod_{j=1}^{n}(\xi_{\nu_j} + i\rho_{\nu_j}) \prod_{j=1}^{\ell}(2\xi_{x_j}\,\rho_{x_j} + i1) \times \prod_{j=1}^{m} \frac{1}{i}(\xi_{\mu_j} - i\rho_{\mu_j}) \exp\left\{ - i\sum_{r=1}^{k} \rho_r(\xi_r - \xi'_r)\right\} d\rho \tag{A.1}$$

if k is even, and

$$\operatorname{Ker}\gamma^{(\nu,x,\mu)}(\xi,\xi') = \int^B \prod_{j=1}^{n}(\xi_{\nu_j} + \rho_{\nu_j}) \cdot \prod_{j=1}^{\ell}(1 - 2\xi_{x_j}\,\rho_{x_j}) \times \prod_{j=1}^{m} \frac{1}{i}(\xi_{\mu_j} - \rho_{\mu_j}) \exp\left\{ - \sum_{r=1}^{n} \rho_r(\xi_r + \xi'_r)\right\} d\rho \tag{A.2}$$

if k is odd.

The proof is straightforward, and we omit it here.

Remark. Here we expressed elements of the algebra $\mathbf{G}_0(k)$ in the so-called 'Fourier-transform' form. In fact, we just multiply the initial expression by ρ-monomials and then integrate over ρ.

Let us set

$$L = \sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)}\,\gamma^{(\nu,x,\mu)}$$

where $\gamma^{(\nu,x,\mu)} \in \mathbf{C}(2k)$, and are of the form (3.8), and $h_{(\nu,x,\mu)} \in \mathbb{C}$.

After computation using (A.1) and (A.2) we obtain

$$\begin{aligned}\operatorname{Ker}(\mathrm{e}^{\frac{1}{n}L})(\xi,\xi') &= \operatorname{Ker}\left(1 - \frac{1}{n}L\right)(\xi,\xi') + \mathrm{o}\left(\frac{1}{n}\right) \\ &= \int^B \exp\left\{ - \frac{1}{n}\sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)} \prod_{\nu}(\xi_\nu + i\rho_\nu)\prod_{x}(2\xi_x\rho_x + i1)\right. \\ &\quad \left. \times \prod_{\mu}\frac{1}{i}(\xi_\mu - i\rho_\mu) - \sum_{r=1}^{n}\rho_r(\xi_r + \xi'_r)\right\} d\rho_n\cdots d\rho_1 + \mathrm{o}\left(\frac{1}{n}\right)\end{aligned}$$

for k even; an analogous formula holds for k odd.

Remark. Since k is fixed, we always understand convergence as a pointwise convergence in a finite-dimensional linear space.

Finally, applying Trotter's formula, we have

$$
\begin{aligned}
\mathrm{e}^{tL} f(\xi) = \lim_{m\to\infty} \int^B \cdots \int^B & \operatorname{Ker}(\mathrm{e}^{(t/m)L}) \left[\xi, \xi\left(\frac{(m-1)t}{m}\right)\right] \\
& \times \operatorname{Ker}(\mathrm{e}^{(t/m)L}) \left[\xi\left(\frac{(m-1)t}{m}\right), \xi\left(\frac{(m-2)t}{m}\right)\right] \\
& \cdots \operatorname{Ker}(\mathrm{e}^{-(t/m)L}) \left[\xi\left(\frac{t}{m}\right), \xi'\right] f(\xi)\, d\xi'\, d\xi\left(\frac{t}{m}\right) \cdots d\xi\left(\frac{(m-1)t}{m}\right) \\
= \lim_{m\to\infty} & \int^B \cdots \int^B \exp\left\{ \sum_{j=1}^{m} \left(-\frac{j\cdot t}{m} \sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)} \prod_{\nu} (\xi_\nu + i\rho_\nu) \right.\right. \\
& \left. \times \prod_{x} (2\xi_x\rho_x + i1) \cdot \prod_{\mu} \frac{1}{i} (\xi_\mu - i\rho_\mu) \right) \\
& \left. - \sum_{j=1}^{m} \sum_{r=1}^{n} \rho_r \left(\frac{j\cdot t}{m}\right) \left(\xi_r \left(\frac{j\cdot t}{m}\right) + \xi_r \left(\frac{(j-1)t}{m}\right)\right)\right\} f(\xi') \\
& \times d\rho\left(\frac{t}{m}\right) \cdots d\rho(t)\, d\xi'\, d\xi\left(\frac{t}{m}\right) \cdots d\xi(t) \\
= \lim_{m\to\infty} & \int^B Q_t^m(\xi, \xi') f(\xi')\, d\xi'
\end{aligned}
\tag{A.3}
$$

from which we obtain theorem 3.1.

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